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Line Graphs and Forbidden Induced Subgraphs

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Beineke and Robertson independently characterized line graphs in terms of nine forbidden induced subgraphs. In 1994, Šoltés gave another characterization, which reduces the number of forbidden induced subgraphs to seven, with only five exceptional cases. A graph is said to be *a dumbbell* if it consists of two complete graphs sharing exactly one common edge. In this paper, we show that a graph with minimum degree at least seven that is not a dumbbell is a line graph if and only if it [illegible]

tions of line graphs and to hamiltonian line graphs are also discussed. © 2001 Academic Press

1. INTRODUCTION

Graphs considered in this paper are simple and finite graphs. We use [4] as a source for undefined terms and notations. For graphs G and H, write $G \cong H$ to mean that the graphs G and H are isomorphic.

Let H be a graph with $E(H) \neq \emptyset$, the *line graph* of H, denoted $L(H)$, is a graph whose vertex set $V(L(H))$ is $E(H)$, where two vertices in $L(H)$ are adjacent if and only if the corresponding edges are adjacent in H. If a graph G is isomorphic to line graph of some graph H, then we simply say that G is a line graph. Given a set of graphs S, we say that a graph G is *S-free* if G contains no induced subgraph isomorphic to a graph in the set S.

The classical results on line graphs are surveyed in Hemminger and Beineke [14] and the recent results in Prisner [17]. One of the major results on line graphs, independently obtained by Beineke [2] and Robertson [18] (see also [12, p. 74]), is the following fundamental theorem.

THEOREM 1.1 [2, 18]. *a connected graph is a line graph if and only if it is $\{G_1, ..., G_9\}$-free, where the set of nine forbidden induced subgraphs $\{G_1, ..., G_9\}$ can be found in Fig.* 1.

Throughout this paper, the notations $G_1, G_2, ..., G_9$ are exclusively used for these graphs shown in Fig. 1.

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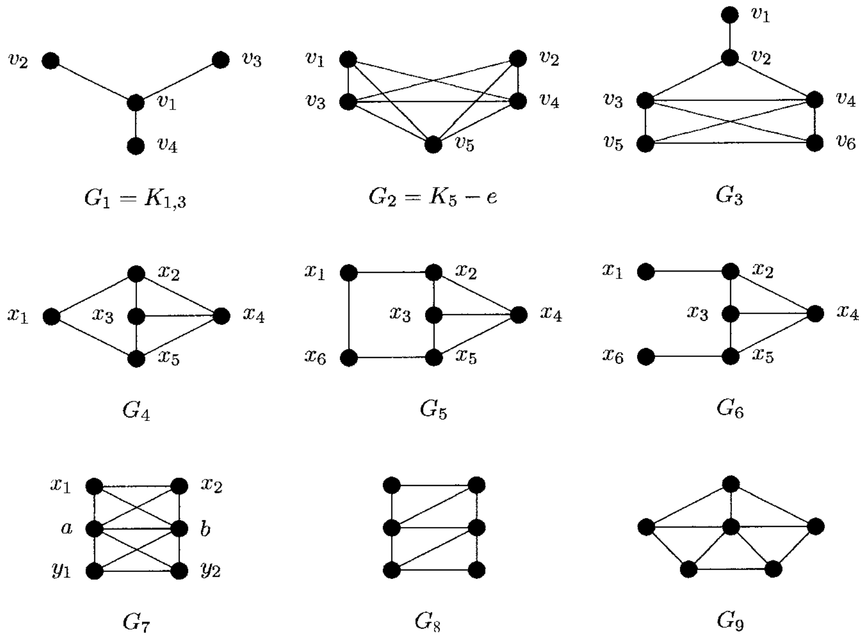


FIG. 1. The minimal forbidden induced subgraphs for line graphs.

Among other results, Bermond and Meyer [3] characterized line graphs of multigraphs by forbidding a set of seven induced subgraphs. Cvetković *et al.* [9] characterized generalized line graphs by forbidding a set of 31 induced subgraphs. Chartrand [8] and Hedetniemi [13] provided a forbidden induced subgraph characterization of line graphs of bipartite graphs and Cai *et al.* [7] characterized line graphs of bipartite multigraphs.

In 1994, Šoltés [20] showed that connected line graphs with at least nine vertices can be characterized by forbidding seven induced subgraphs.

THEOREM 1.2 [20]. *A connected graph is a line graph if and only if it is* $\{G_1, ..., G_7\}$*-free and nonisomorphic to any of the graphs* G_8, G_9, H_1, H_2, *and* H_3 (*see Fig.* 2 *for graphs* H_1, H_2, *and* H_3).

The goal of this paper is to further reduce the number of necessary forbidden induced subgraphs by increasing the connectivity and the minimum degree of the graph in question. Let $\Lambda_{k,\delta}$ denote the set of k-connected line graphs with minimum degree at least δ. We say that $\Lambda_{k,\delta}$ *can be characterized by the graphs* $J_1, ..., J_n$ if the following statement holds.

> A k-connected graph G with minimum degree at least δ is a line graph if and only if G is $\{J_1, ..., J_n\}$-free.

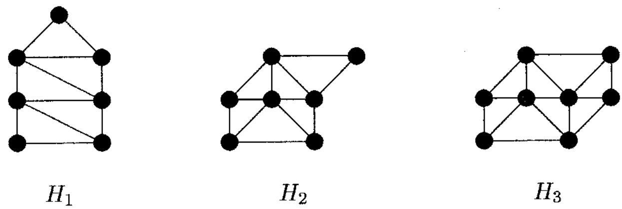


FIG. 2. The three graphs that contain G_8 or G_9 as a proper induced subgraph and do not contain any of the graphs $G_1, ..., G_7$.

A graph is said to be *a dumbbell* if it consists of two complete graphs sharing exactly one common edge. The following are our main results and its corollaries.

THEOREM 1.3. *A graph with minimum degree at least seven that is not a dumbbell is a line graph if and only if it is* $\{K_{1,3}, K_5 - e, G_3\}$*-free.*

COROLLARY 1.4. *A* 3*-connected graph with minimum degree at least seven is a line graph if and only if its is* $\{K_{1,3}, K_5 - e, G_3\}$*-free.*

COROLLARY 1.5. *For any integer* $k \geqslant 1$ *and* $\delta \geqslant k$*, none of the sets* $\Lambda_{k,\delta}$ *can be characterized by two graphs. Moreover,* $\Lambda_{k,\delta}$ *can be characterized by three graphs if and only if it is a subset of the set* $\Lambda_{3,7}$.

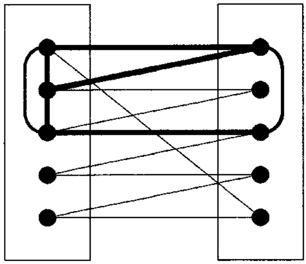
COROLLARY 1.6. *A connected graph with the minimum degree at least six that is not a dumbbell is a line graph if and only if it is* $\{K_{1,3}, K_5 - e, G_3, G_4\}$*-free.*

COROLLARY 1.7. *A connected graph that is not a dumbbell is a line graph if and only if it is* $\{K_{1,3}, K_5 - e, G_3, G_4, G_5, G_6\}$*-free and nonisomorphic to any of* G_8, G_9, H_1, H_2, *and* H_3.

Figure 3 shows a 6-connected 6-regular $\{K_{1,3}, K_5 - e, G_3\}$-free graph that is not a line graph. Therefore Corollary 1.4 cannot be improved by decreasing the minimum degree.

Corollary 1.5 shows that the number of forbidden induced subgraphs in Theorem 1.3 and Corollary 1.4 cannot be reduced and that $\Lambda_{3,7}$ is the largest set of the form $\Lambda_{k,\delta}$ that can be characterized by this minimal number of graphs.

For presentational convenience, we also use the following notations and terminology. Given a sequence of vertices $v_1, v_2, ..., v_n$ of a graph G, $\langle v_1, ..., v_n \rangle$ denotes the subgraph induced by $\{v_1, ..., v_n\}$. An induced subgraph isomorphic to $K_{1,3}$ is also called a *claw*, with the only vertex of degree three being the *center* of the claw. A graph H isomorphic to one of



H_4

FIG. 3. A 6-connected 6-regular $\{K_{1,3}, K_5-e, G_3\}$-free graph consisting of two copies of K_5 joined by a 10-cycle. The thick subgraph is isomorphic to G_4, hence the graph is not a line graph. The rectangles represent complete subgraphs.

$\{G_1, G_2, G_3, G_4, G_5, G_6\}$ will represented by listing its vertices in a sequence, with the following rule:

(i) If $H=G_1$, then the only vertex of degree 3 is always the first vertex in the sequence.

(ii) If $H=G_2$, then the first two vertices in the sequence will be the two nonadjacent vertices in G_2.

(iii) The vertices of copies of G_3, G_4, G_5, and G_6 are always listed in the order indicated in Fig. 1.

Fix a specified G_i, $1\leqslant i\leqslant 9$. Let G be a graph which contains G_i as an induced subgraph. For a vertex $v\in V(G)$, $N(v)$ denotes the set of vertices in $V(G)-V(G_i)$ that are adjacent to v in G; and for an edge $e=ab\in E(G)$, denote $N(e)=N(a)\cap N(b)$. For a vertex $v\in V(G)-V(G_i)$, the *trace* of v, denoted $tr(v)$, is the set of vertices of G_i adjacent to v.

In separated sections that follow, we present the proof for Theorem 1.3 by showing how each of the graphs in $\{G_4, ..., G_7\}$ can be excluded from the list of forbidden induced subgraph when additional minimum degree conditions are imposed. Then, we prove the main result and the corollaries. The last section is devoted to application to hamiltonian line graphs.

2. EXCLUDING G_7

THEOREM 2.1. *Let G be a $\{K_{1,3}, K_5-e, G_3, G_4\}$-free connected graph. Then either G is a dumbbell or G is G_7-free.*

Proof. Suppose that the graph G contains a copy of $G_7=\langle a, b, y_1, y_2, x_1, x_2\rangle$ as in Fig. 1. We will show that G is a dumbbell consisting of two complete subgraphs sharing the edge ab. Let Y be the vertex set of a maximal clique in G containing vertices a, b, y_1, and y_2 and let X be the vertex set of a maximal clique containing vertices a, b, x_1, and x_2.

CLAIM 1. *If $x \in X \setminus \{a, b\}$ and $y \in Y \setminus \{a, b\}$ then x and y are not adjacent.*

Proof of Claim 1. Suppose that x and y are adjacent. Since G_7 is an induced subgraph of G, it cannot contain both x and y. Without loss of generality assume that $x \notin V(G_7)$. If $yx_1 \notin E(G)$ then $\langle y, x_1, x, a, b \rangle \cong K_5 - e$. Hence $yx_1 \in E(G)$, thus $y \notin V(G_7)$.

Similarly $xy_1 \in E(G)$, otherwise $\langle x, y_1, y, a, b \rangle \cong K_5 - e$. It follows that $\langle x_1, y_1, x, y, a \rangle \cong K_5 - e$, a contradiction. ∎

CLAIM 2. $V(G) = X \cup Y$.

Proof of Claim 2. Suppose to the contrary that G contains a vertex v that is not in $X \cup Y$. Since G is connected, the vertex v is adjacent to some vertex in $X \cup Y$. We will distinguish two cases.

Case 1. Suppose that v is adjacent to a or b. Without loss of generality assume that v is adjacent to a. If there are vertices $x \in X \setminus \{a, b\}$ and $y \in Y \setminus \{a, b\}$ such that vx and vy are non-edges, then by Claim 1, xy is a non-edge, thus the vertices a, v, x, and y induce a claw. Hence either v is adjacent to all vertices in $X \setminus \{a, b\}$, whence $vb \notin E(G)$, and so $\langle v, b, a, x_1, x_2 \rangle \cong K_5 - e$, a contradiction; or v is adjacent to all vertices in $Y \setminus \{a, b\}$, whence $vb \in E(G)$ and so $v \in X$, contrary to the assumption that $v \notin X \cup Y$.

Case 2. Suppose that v is adjacent to neither a nor b. Then without loss of generality we can assume that $vx_1 \in E(G)$. Then either $vy_1 \in E(G)$ or $vy_2 \in E(G)$; for otherwise $\langle v, x_1, a, b, y_1, y_2 \rangle \cong G_3$. Thus we may assume that $vy_1 \in E(G)$. Then $\langle v, y_1, a, b, x_1 \rangle \cong G_4$, a contradiction. ∎

Thus G must be a dumbbell, and so the proof of Theorem 2.1 is complete. ∎

3. EXCLUDING G_6

THEOREM 3.1. *Each $\{K_{1,3}, K_5 - e, G_3\}$-free graph of minimum degree at least six is G_6-free.*

The icosahedron is a 5-regular $\{K_{1,3}, K_5 - e, G_3\}$-free graph containing a copy of G_6, thus Theorem 3.1 is best possible in the sense that the minimum degree condition cannot be relaxed. We need a lemma before the proof.

LEMMA 3.2. *Let G be a $\{K_{1,3}, K_5-e\}$-free graph containing the wheel $W_5=G_9$ as an induced subgraph. Then the degree of the central vertex of W_5 in G is five.*

Proof. Assume that the copy of W_5 in G consists of the cycle $C_5=a_1a_2a_3a_4a_5a_1$ and the central vertex c. Suppose to the contrary that the degree of c is at least six. Hence c has a neighbor v that is not in the wheel W_5. Then v is adjacent to a vertex of C, say a_1, otherwise $\langle c, a_1, a_3, v\rangle$ is a claw. To avoid the claw $\langle c, a_2, a_5, v\rangle$, the vertex v must be adjacent to a_2 or a_5, say $va_2 \in E(G)$. To avoid the claw $\langle c, v, a_3, a_5\rangle$ the vertex v must be adjacent to a_3 or a_5, say $va_3 \in E(G)$. Now $\langle a_1, a_3, a_2, c, v\rangle \cong K_5-e$, a contradiction. ▮

Proof of Theorem 3.1. Assume to the contrary that a $\{K_{1,3}, K_5-e, G_3\}$-free graph G of minimum degree at least six contains the graph G_6 as an induced subgraph. The vertices of G_6 are labeled $x_1, x_2, ..., x_6$ as in Fig. 1. For $i \leqslant 6$, let N_i denote the set of vertices in $V(G)-V(G_6)$ adjacent to the vertex x_i. We will reach a contradiction by showing that some vertex in G has degree at most five.

We start with characterizations of the traces of vertices adjacent to x_3 or x_4. By the symmetries of the graph G_6, and to avoid a potential claw centered at x_3 containing x_2 and x_5, each vertex in N_3 must be adjacent to x_2 or x_5. These two cases are symmetric, therefore it suffices to study traces of the vertices in $N_2 \cap N_3$.

CLAIM 3. *For each $v \in N_2 \cap N_3$, either $v \in N_1$ and $tr(v)=\{x_1, x_2, x_3\}$ or $v \notin N_1$ and $tr(v)=\{x_2, x_3, x_4, x_6\}$.*

Proof of Claim 3. If $v \in N_1 \cap N_2 \cap N_3$, then $tr(v)=\{x_1, x_2, x_3\}$, as each of the following assertions must hold.

(i) $v \notin N_6$. Otherwise $\langle v, x_1, x_3, x_6\rangle$ is a claw.

(ii) $v \notin N_4$. Otherwise either $v \in N_5$ whence $\langle x_2, x_5, v, x_3, x_4\rangle \cong K_5-e$; or $v \notin N_5$ whence $\langle x_6, x_5, x_4, x_3, x_2, v\rangle \cong G_3$. Contradictions obtain in either cases.

(iii) $v \notin N_5$. Otherwise $\langle x_5, v, x_4, x_6\rangle$ is a claw.

If $v \in (N_2 \cap N_3)\backslash N_1$, then $tr(v)=\{x_2, x_3, x_4, x_6\}$, as each of the following assertions must hold.

(iv) $v \in N_4$; otherwise $\langle x_2, x_1, v, x_4\rangle$ is a claw.

(v) $v \notin N_5$; otherwise $\langle x_2, x_5, v, x_3, x_4\rangle \cong K_5-e$.

(vi) $v \in N_6$; otherwise $\langle x_6, x_5, x_4, x_3, x_2, v\rangle \cong G_3$. ▮

For each edge $x_i x_j \in E(G_6)$, let $N_{ij} = N(x_i x_j)$. Let C be the four-cycle $x_2 x_3 x_5 x_4 x_2$. by Claim 3, the trace of a vertex in N_{23} is either $\{x_1, x_2, x_3\}$ or $\{x_2, x_3, x_4, x_6\}$. By the symmetries of G_6, we can similarly characterize the possible traces of the vertices in $N(e)$, for any edge $e \in E(C)$. For example, the trace of a vertex in N_{45} is either $\{x_4, x_5, x_6\}$ or $\{x_3, x_4, x_5, x_1\}$.

A vertex v in $N_3 \cup N_4$ is said to be of *type three* if $|tr(v)| = 3$ and of *type four* if $|tr(v)| = 4$. The following claims enlighten the structure of the subgraph induced by $V(G_6) \cup N_3 \cup N_4$. Since G is claw-free, $N_3 = N_{23} \cup N_{35}$ and $N_4 = N_{24} \cup N_{45}$. By Claim 3, every vertex in $N_3 \cup N_4$ is either of type 3 or of type 4.

CLAIM 4. *Let e be edge in the cycle $C = x_2 x_3 x_5 x_4 x_2$. Then each of the following holds.*

(1) *$N(e)$ induces a complete graph.*

(2) *All the vertices in $N(e)$ are of the same type.*

(3) *$N(e)$ has at most one vertex of type three.*

(4) *$N(e)$ has at most two vertices of type four.*

Proof of Claim 4. Without loss of generality we can assume that $e = x_2 x_3$.

(1) If $v_1, v_2 \in N(e)$, then by Claim 3, both $v_1, v_2 \in N(x_3)$ and none of them is adjacent to x_5. Thus v_1 is adjacent to v_2, otherwise $\langle x_3, x_5, v_1, v_2 \rangle$ is a claw.

(2) Suppose that v_1 is a vertex of type three, v_2 is a vertex of type four and both are in N_{23}. Then $tr(v_1) = \{x_1, x_2, x_3\}$ and $tr(v_2) = \{x_2, x_3, x_4, x_6\}$. By Claim 4(1), $v_1 v_2 \in E(G)$, and so $\langle v_2, v_1, x_4, x_6 \rangle \cong K_{1,3}$, a contradiction.

(3) Suppose that v_1 and v_2 are two vertices in N_{23} of type three. Then $tr(v_1) = tr(v_2) = \{x_1, x_2, x_3\}$. By Claim 4(1), $v_1 v_2 \in E(G)$, and so $\langle x_1, x_3, x_2, v_1, v_2 \rangle \cong K_5 - e$, a contradiction.

(4) Suppose that v_1, v_2, and v_3 are three vertices of type four in N_{23}. Thus $tr(v_1) = tr(v_2) = tr(v_3) = \{x_2, x_3, x_4, x_6\}$. By Claim 4(1), the vertices v_1, v_2 and v_3 induce a triangle, and so $\langle x_6, x_3, v_1, v_2, v_3 \rangle \cong K_5 - e$. ∎

CLAIM 5. *The set $N_3 \cup N_4$ contains at most two vertices of type four.*

Proof of Claim 5. Assume to the contrary that $u_1, u_2, v \in N_3 \cup N_4$ and three vertices of type four. Note that a vertex of type four belongs either to both N_{23} and N_{24} or to both N_{35} and N_{45}. By Claim 4(4), each of N_{23} and N_{45} contains at most two vertices of type four. Thus without loss of generality we assume that both $u_1, u_2 \in N_{23}$ and $v \in N_{45}$. This yields $tr(u_1) = tr(u_2) = \{x_2, x_3, x_4, x_6\}$ and $tr(v) = \{x_1, x_3, x_4, x_5\}$.

By Claim 4(3), we have $u_1 u_2 \in E(G)$. If v is adjacent to u_1 or u_2, say to u_1, then $\langle v, x_1, u_1, x_5 \rangle$ is a claw. Hence $vu_1, vu_2 \notin E(G)$, and so $\langle x_1, v, x_4, x_3, u_1, u_2 \rangle = G_3$, a contradiction. ∎

CLAIM 6. *The set* $N_{23} \cup N_{24}$ *contains at most one vertex of type three. Similarly, the set* $N_{35} \cup N_{45}$ *contains at most one vertex of type three. Therefore* $N_3 \cup N_4$ *contains at most two vertices of type three.*

Proof of Claim 6. By the symmetries of G_6, it suffices to prove the first statement. By contradiction, assume that $N_{23} \cup N_{24}$ contains two vertices u and v of type three. By Claim 4(3), each of N_{23} and N_{24} contains at most one of u and v. Thus we may assume that $u \in N_{23}$ and $v \in N_{24}$. It follows that $tr(u) = \{x_1, x_2, x_3\}$ and $tr(v) = \{x_1, x_2, x_4\}$. If $uv \in E(G)$ then $\langle x_5, x_4, v, x_2, x_1, u \rangle \cong G_3$, a contradiction. Thus $uv \notin E(G)$, and so $\langle x_2, v, x_4, x_3, u, x_1 \rangle$ is isomorphic to the wheel W_5. By Lemma 3.2, the degree of x_2 in G is five, a contradiction. ∎

To conclude the proof, suppose first that $|N_3 \cup N_4| \geqslant 3$. By Claim 6, and by $N_3 \cup N_4 = N_{23} \cup N_{35} \cup N_{24} \cup N_{45}$, at most two vertices in $N_2 \cap N_4$ are of type three, and so we may assume that there exists $v \in N_{23} \cap N_{24}$ which is of type four. Since by Claim 4(2) all the vertices in N_{23} (and in N_{24}, respectively) are of the same type, the set $N_{23} \cup N_{24}$ contains no vertex of type three. Then by Claims 5 and 6 the set $N_3 \cup N_4$ contains at most one vertex of type three and at most two of type four. Hence $|N_3 \cup N_4| \leqslant 3$ with equality only if (up to isomorphism) N_{35} contains a vertex of type three and N_{23} contains two vertices of type four, and so $\deg(x_4) \leqslant 5$, a contradiction.

Hence $|N_3 \cup N_4| \leqslant 2$. This again yields $\deg(x_4) \leqslant 5$, a contradiction. ∎

4. EXCLUDING G_5

THEOREM 4.1. *Each* $\{K_{1,3}, K_5 - e, G_3\}$*-free graph of minimum degree at least six is* G_5*-free.*

Proof. Suppose to the contrary that a $\{K_{1,3}, K_5 - e, G_3\}$-free graph G of minimum degree at least six contains a copy of G_5 with the vertices labeled as $x_1, ..., x_6$ (see Fig. 1). Let $N_i = N(x_i)$ and $N_{ij} = N(x_i) \cap N(x_j)$, for all i, j with $1 \leqslant i, j \leqslant 6$.

CLAIM 7. *Each vertex in N_{23} is adjacent to x_1 and nonadjacent to x_5.*

Proof of Claim 7. Let $v \in N_{23}$.
If $vx_1 \notin E(G)$, then we must have each of the following:

$vx_4 \in E(G)$, otherwise $\langle x_2, v, x_1, x_4 \rangle$ is a claw;
$vx_5 \notin E(G)$, otherwise $\langle x_5, x_2, x_3, x_4, v \rangle \cong K_5 - e$;
$vx_6 \in E(G)$, otherwise $\langle x_6, x_5, x_4, x_3, x_2, v \rangle \cong G_3$.

But then, $\langle x_6, x_1, v, x_5 \rangle$ is a claw, a contradiction. Hence we must have $vx_1 \in E(G)$.
If $vx_5 \in E(G)$, then each of the following must hold:

$vx_4 \notin E(G)$, otherwise $\langle x_5, x_2, x_3, x_4, v \rangle \cong K_5 - e$;
$vx_6 \in E(G)$, otherwise $\langle x_5, v, x_4, x_6 \rangle$ is a claw.

It follows that v is the center of the wheel $\langle v, x_1, x_2, x_3, x_5, x_6 \rangle$. By Lemma 3.2, the degree of v is five, a contradiction. This proves Claim 7. ∎

Since $\deg(x_3) \geqslant 6$, the set $N_3 = N_{23} \cup N_{35}$ contains at least three vertices. Without loss of generality we assume that N_{23} contains two distinct vertices u and v. By Claim 7, u and v are adjacent to x_1 and nonadjacent to x_5. If $uv \notin E(G)$ then $\langle x_3, u, v, x_5 \rangle$ is a claw. Otherwise $\langle x_1, x_3, u, v, x_2 \rangle \cong K_5 - e$, a contradiction. This completes the proof of Theorem 4.1. ∎

5. EXCLUDING G_4

THEOREM 5.1. *A $\{K_{1,3}, K_5 - e, G_3\}$-free graph with the minimum degree at least seven is G_4-free.*

Figure 3 shows a 6-regular $\{K_{1,3}, K_5 - e, G_3\}$-free graph containing a copy of G_4. Hence Theorem 5.1 is best possible in the sense that the minimum degree condition cannot be relaxed.

Proof. By contradiction, assume that a $\{K_{1,3}, K_5 - e, G_3\}$-free graph G with minimum degree at least seven contains a copy of G_4 with the vertices labelled $x_1, ..., x_5$ as in the Fig. 1. Let N_i be the set of the vertices not in G_4 that are adjacent to x_i $(i \leqslant 5)$. Set $N_{ij} = N_i \cap N_j$ for $i, j \in \{1, 2, 3, 4, 5\}$ and let $tr(v)$ be the set of all neighbors of the vertex v belonging to G_4.

CLAIM 8. *The trace of a vertex in N_{23} is one of the following four sets: $A_1 = \{x_1, x_2, x_3\}$, $A_2 = \{x_1, x_2, x_3, x_5\}$, $A_3 = \{x_1, x_2, x_3, x_4\}$, and $A_4 = \{x_2, x_3, x_4\}$.*

Proof of the Claim 8. Let $v \in N_{23}$. First observe that if $v \in N_{45}$ then $\langle x_2, x_5, x_3, x_4, v \rangle \cong K_5 - e$. Thus $v \notin N_4$ or $v \notin N_5$.

If, in addition, $v \in N_1$ then the previous statement implies that the trace of v is one of the sets A_1, A_2, or A_3. If $v \notin N_1$ then $v \in N_4$, otherwise $\langle x_2, v, x_1, x_4 \rangle$ is a claw. Since $v \in N_4$, we have $v \notin N_5$. Therefore $tr(v) = A_4 = \{x_2, x_3, x_4\}$. ∎

Let C be the four-cycle $x_2x_3x_5x_4x_2$. For an l with $1 \leqslant l \leqslant 4$, call a vertex $v \in N(e)$ for an edge $e \in E(C)$ *of type* (A_l, e), or shortly A_l, if $tr(v) = p(A_l)$, where p is the automorphism of G_4 that maps the edge x_2x_3 onto e. Let $N^l(e)$ be the set of vertices in $N(e)$ of type (A_l, e). If $e = x_ix_j$ then set $N^l_{ij} = N^l(x_ix_j)$.

Since G is claw-free, $N_3 = N_{23} \cup N_{35}$ and $N_4 = N_{24} \cup N_{45}$. Therefore every vertex in $N_3 \cup N_4$ is of one of the four types A_1, A_2, A_3, and A_4.

CLAIM 9. *For any edge e in the cycle C we have*

(1) *The set* $N^1(e) \cup N^2(e) \cup N^3(e)$ *contains at most two vertices. Moreover, if it contains two vertices then they are not adjacent and one is in* $N^2(e)$ *and the other one in* $N^3(e)$.

(2) $|N^4(e)| \leqslant 1$.

(3) $|N(e)| \leqslant 3$, *and the equality holds only if the vertices in* $N(e)$ *are of the types* A_2, A_3, *and* A_4.

(4) *Each of the sets* $N^l(e)$ $(l = 1, 2, 3, 4)$ *contains at most one element.*

(5) *The set* $N^l(e)$ *is empty.*

Proof of Claim 9. Without loss of generality we can assume that $e = x_2x_3$.

(1) Let $v_1, v_2 \in N^1(e) \cup N^2(e) \cup N^3(e)$. If $v_1v_2 \in E(G)$, then $v_ix_j \in E(G)$, for $1 \leqslant i \leqslant 2$ and $1 \leqslant j \leqslant 3$. Thus $\langle x_1, x_3, x_2, v_1, v_2 \rangle \cong K_5 - e$, a contradiction.

Hence $v_1v_2 \notin E(G)$. At least one of them, say v_1, must be adjacent to x_5, otherwise $\langle x_3, x_5, v_1, v_2 \rangle$ is a claw. Thus $v_1 \in N^2(e)$. Similarly at least one of them is adjacent to x_4, otherwise $\langle x_3, x_4, v_1, v_2 \rangle$ is a claw. By the definition of A_1, A_2, and A_3, one has $v_2 \in N^3(e)$, and so the set $N^1(e) \cup N^2(e) \cup N^3(e)$ contains at most two vertices.

(2) Let $v_1, v_2 \in N^4(e)$. If $v_1v_2 \notin E(G)$, then $\langle v_1, v_2, x_2, x_3, x_4 \rangle \cong K_5 - e$. If $v_1v_2 \in E(G)$, then $\langle x_1, x_5, x_4, x_3, v_1, v_2 \rangle \cong G_3$, and so a contradiction arises in either case. Therefore $N^4(e)$ contains at most one vertex.

(3) and (4) These follow from (1) and (2).

(5) Assume that there exists a vertex $v \in N^1(e)$. First we show that $N(e) = N^1(e) = \{v\}$. If not, then there is a vertex $v_1 \in N(e)\backslash\{v\}$. By Claim 9(1), $v_1 \in N^4(e)$. If $vv_1 \notin E(G)$, then $\langle x_3, x_5, v, v_1 \rangle$ is a claw. If $vv_1 \in E(G)$, then $\langle x_4, v, v_1, x_2, x_3 \rangle \cong K_5 - e$. Thus $N_{23} = N^1_{23} = \{v\}$.

By $|N(x_3) \cap (V(G_4) \cup \{v\})| = 4$ and by $|N(x_3)| \geqslant 7$, there exist vertices $v_2, v_3, v_4 \in N(x_3) \backslash V(G_4) \cup \{v\}$. Since $N_{23} = \{v\}$, one concludes that $v_2, v_3,$ and $v_4 \notin N(x_2)$. Therefore each $x_5 v_i \in E(G)$ for each $i = 2, 3, 4$, for otherwise $\langle x_3, x_2, x_5, v_i \rangle$ is a claw. It follows that $\{v_2, v_3, v_4\} \subseteq N_{35}$. By Claim 9(3), for $e = x_3 x_5$, we may assume that $v_2 \in N_{35}^2$, $v_3 \in N_{35}^3$ and $v_4 \in N_{35}^4$. Since $N_{35}^2 = N_{23}^2$, one has $v_2 \in N_{23}^2$, contrary to the fact that $N_{23}^2 = \emptyset$. Thus N_{23}^1 must be empty. ∎

Claim 10. *The set of $N_3 \cup N_4$ contains no vertex of type A_2.*

Proof of Claim 10. By contradiction, assume that $N_3 \cup N_4$ contains a vertex v_2 of type A_2. Say, $v_2 \in N_{23}$ and $tr(v_2) = \{x_1, x_2, x_3, x_5\}$.

First we show that there is no other vertex of type A_2 in $N_3 \cup N_4$. Since $N_{23}^2 = N_{35}^2 = \{v_2\}$ by Claim 9(4) and by $N_{24}^2 = N_{45}^2$, if $u_2 \in N_3 \cup N_4$ is another vertex of type A_2 then $u_2 \in N_{24}^2$ and the trace of u_2 is $\{x_1, x_2, x_4, x_5\}$.

If $u_2 v_2 \in E(G)$, then $\langle x_2, x_5, x_1, v_2, u_2 \rangle \cong K_5 - e$. Hence $u_2 v_2 \notin E(G)$ and so one has $\langle x_2, v_2, x_1, u_2, x_4, x_3 \rangle \cong W_5$. By Lemma 3.2, deg $(x_2) = 5$ in G, contrary to the assumption that every vertex of G has degree at least seven. Thus we have proved that v_2 is the only vertex in $N_3 \cup N_4$ of type A_2. By Claim 9(5), $N_3 \cup N_4$ contains no vertex of type A_1, and so all its vertices except for v_2 are of type A_3 or A_4. We say that a vertex outside G_4 is *symmetric* if it is of type A_3 or A_4.

Suppose that both N_{23} and N_{35} contain a symmetric vertex. Let v be a symmetric vertex in N_{23}, thus $tr(v) = \{x_1, x_2, x_3, x_4\}$ or $tr(v) = \{x_2, x_3, x_4\}$. Let u be a symmetric vertex in N_{35}, thus its trace is either $\{x_1, x_3, x_4, x_5\}$ or $\{x_3, x_4, x_5\}$. Then each of the following must hold:

(i) $vu \notin E(G)$, otherwise $\langle x_2, u, v, x_3, x_4 \rangle \cong K_5 - e$;

(ii) $v_2 v \notin E(G)$, otherwise $\langle x_4, v_2, v, x_2, x_3 \rangle \cong K_5 - e$;

(iii) $v_2 u \notin E(G)$, otherwise $\langle x_4, v_2, u, x_3, x_5 \rangle \cong K_5 - e$.

Therefore $\langle x_3, v, u, v_2 \rangle \cong K_{1,3}$, a contradiction. It follows that we may assume that at least one of N_{23} and N_{35} does not contain a symmetric vertex.

Assume that N_{23} does not contain a symmetric vertex. Then $N_{23} = \{v_2\}$ and $N_{24} = \emptyset$ since each symmetric vertex in N_{24} would belong to N_{23}. Moreover the set $N_{35} \cup N_{45}$ contains only symmetric vertices, therefore $N_{35} = N_{45}$. By Claim 9(4) the set $N_{35} \cup N_{45}$ contains at most one vertex of type A_3 and at most one vertex of type A_4. Then $|N_4| = |N_{24} \cup N_{45}| \leqslant 0 + 2 = 2$. Hence deg $(x_4) \leqslant 5$, a contradiction.

Hence N_{35} contains no symmetric vertices. Thus N_{34} contains no symmetric vertex either and both N_{45} and N_{34} are empty. The set N_{24} contains at most two vertices (at most one of type A_3 and at most one of type A_4). Thus deg $(x_4) \leqslant 5$, a contradiction. The claim is proved. ∎

Now we are in a position to finish the proof of the theorem. From the last two claims we know that all vertices in $N_3 \cup N_4$ are of type A_3 or A_4. Since $N^l_{23} = N^l_{24}$ and $N^l_{35} = N^l_{45}$ for $l = 3, 4$, we have

$$N_3 \cup N_4 = N_{23} \cup N_{35} \cup N_{24} \cup N_{45} = N^3_{23} \cup N^4_{23} \cup N^3_{35} \cup N^4_{35}. \qquad (1)$$

Thus $|N_3 \cup N_4| \leqslant 4$, by Claim 9(4). Since the vertex x_3 has three neighbors in G_4 and at least seven neighbors in G, it follows that $|N_3 \cup N_4| = 4$. By (1), two vertices in $N_3 \cup N_4$ are of type A_3 and two others of type A_4. Hence $N_3 \cup N_4 = \{v_1, v_2, u_1, u_2\}$ with $tr(v_1) = A_3 = \{x_1, x_2, x_3, x_4\}$, $tr(v_2) = A_4 = \{x_2, x_3, x_4\}$, $tr(u_1) = \{x_1, x_3, x_4, x_5\}$, and $tr(u_2) = \{x_3, x_4, x_5\}$. Note that each of the following must hold:

(iv) $v_1 v_2 \in E(G)$, otherwise $\langle v_1, v_2, x_2, x_3, x_4 \rangle \cong K_5 - e$;

(v) $u_1 u_2 \in E(G)$, otherwise $\langle u_1, u_2, x_3, x_4, x_5 \rangle \cong K_5 - e$;

(vi) $u_i v_j \notin E(G)$ for $(i, j \in \{1, 2\})$, otherwise $\langle x_2, u_i, v_j, x_3, x_4 \rangle \cong K_5 - e$.

Therefore the graph induces by $\{x_2, x_3, x_4, x_5\} \cup N_3 \cup N_4$ is a dumbbell consisting of two copies of K_5 sharing the common edge $x_3 x_4$. Let G_0 be the subgraph of G induced by $V(G_4) \cup N_3 \cup N_4$. Since $|N_3 \cup N_4| = 4$, the vertices x_3 and x_4 have no neighbors outside G_0.

Now we will focus our attention on the neighbors of x_1. Since x_1 has four neighbors in G_0 and at least seven in G, there are three vertices y_1, y_2 and y_3 not in G_0 that are adjacent to x_1. Note that the neighborhood of x_1 in graph G_0 consists of two independent edges $e_1 = v_1 x_2$ and $e_2 = u_1 x_5$. Then each vertex y_i, $(i = 1, 2, 3)$ is adjacent to both vertices of the edge e_1 or to both vertices of the edge e_2, otherwise we have a claw with the center x_1 containing y_i and one vertex of both e_1 and e_2.

Since there is an automorphism of G_0 that interchanges the edges e_1 and e_2, we can assume that both vertices y_1 and y_2 are adjacent to both ends of the edge $e_2 = x_5 u_1$. Moreover $y_1 y_2 \in E(G)$, otherwise $\langle x_5, x_4, y_1, y_2 \rangle$ is a claw, since x_3 and x_4 have no neighbors outside G_0. Finally, $y_i u_2 \notin E(G)$ for $i = 1, 2$, otherwise $\langle x_1, u_2, u_1, x_5, y_i \rangle \cong K_5 - e$.

Set $S = \{v_1, v_2, x_2\}$, $S' = \{x_1, y_1, y_2\}$ and $E^0 = \{xy \mid x \in S \text{ and } y \in S'\}$.

Claim 11. *There is a six-cycle C_6 in G such that $E(C_6) = E^0$.*

Proof. Since $|S| = |S'| = 3$, it suffices to prove that each vertex in S part is adjacent to precisely two vertices in S' and vise versa. Let s be a vertex in S. If s is adjacent to all three vertices in S' then $\langle s, x_5, x_1, y_1, y_2 \rangle \cong K_5 - e$. If s is adjacent to at most one vertex in S', then s is non-adjacent to some s_1 and $s_2 \in S'$. Then $\langle s, x_3, u_1, x_5, s_1, s_2 \rangle \cong G_3$. Therefore each vertex in S is adjacent to exactly two vertices in S'. Using a symmetric argument, we conclude that each vertex in S' is adjacent to exactly two vertices in S. This proves the claim. ▮

Let (up to isomorphism) the six-cycle in Claim 11 be $C_6 = x_2 x_1 v_1 y_1 v_2 y_2 x_2$, and let G' be the graph induced by $V(G_0) \cup \{y_1, y_2\}$. Since the degree of each vertex of this C_6 is six, there is a vertex a outside the graph G' that is adjacent to some vertices of C_6. If a is adjacent to all the vertices of C_6 then $\langle x_2, y_1, v_1, v_2, a\rangle \cong K_5 - e$. Therefore we can assume that there are two consecutive vertices on C_6 such that the vertex a is adjacent to only one of them. Without loss of generality we can assume that $ax_2 \in E(G)$ and $ax_1 \notin E(G)$. Then

$$av_2 \in E(G) \text{ otherwise } \langle x_2, a, x_1, v_2\rangle \cong K_{1,3},$$

$$ax_3 \in E(G), \text{ otherwise } \langle x_2, a, x_1, x_3\rangle \cong K_{1,3},$$

$$ax_4 \in E(G), \text{ otherwise } \langle x_2, a, x_1, x_4\rangle \cong K_{1,3},$$

$$av_1 \in E(G), \text{ otherwise } \langle a, v_1, x_2, v_2, x_3\rangle \cong K_5 - e,$$

$$av_1 \notin E(G), \text{ otherwise } \langle u_1, v_2, a, x_3, x_4\rangle \cong K_5 - e.$$

Now $\langle x_1, u_1, x_3, x_4, v_2, a\rangle \cong G_3$, a contradiction. This completes the proof.

6. PROOFS OF THE MAIN RESULTS

It is straightforward to see that Theorem 1.2, 2.1, 3.1, 4.1, and 5.1 together imply Theorem 1.3.

As a 3-connected graph cannot be a dumbbell, Corollary 1.4 follows immediately from Theorem 1.3.

By Theorem 1.1, any line graph must be $\{K_{1,3}, K_5 - e, G_3, G_4\}$-free. Conversely, if G is a connected graph with minimum degree at least six, if G is not a dumbbell, and if G is $\{K_{1,3}, K_5 - e, G_3, G_4\}$-free, then G cannot be any graph in $\{G_8, G_9, H_1, H_2, H_3\}$. By Theorems 2.1, 3.1, and 4.1, G is also $\{G_5, G_6, G_7\}$-free. By Theorem 1.2, G is a line graph, and so Corollary 1.6 obtains.

By Theorem 1.2 and by Theorem 2.1, if a connected graph G is a line graph, then G must be $\{K_{1,3}, K_5 - e, G_3, G_4, G_5, G_6\}$-free and nonisomorphic to any of G_8, G_9, H_1, H_2, and H_3. Conversely, assume that G is not a dumbbell, and that G is $\{K_{1,3}, K_5 - e, G_3, G_4, G_5, G_6\}$-free and nonisomorphic to any of G_8, G_9, H_1, H_2 and H_3. Then by Theorem 2.1, G is also G_7-free, and so by Theorem 1.2, G is a line graph. Hence Corollary 1.7 obtains.

To prove Corollary 1.5, we need two more lemmas.

For a set S of graphs, let $F_{k,\delta}(S)$ denote the set of all k-connected S-free graphs with minimum degree at least δ. Thus

$$\Lambda_{k,\delta} \text{ can be characterized by } S \text{ if and only if } \Lambda_{k,\delta} = F_{k,\delta}(S). \tag{2}$$

If $\Lambda_{k,\delta} = F_{k,\delta}(S)$ and if S is minimum with respect to this property, then S is a *base* of $\Lambda_{k,\delta}$. By Theorem 1.1, $\Lambda_{k,\delta} = F_{k,\delta}(\{G_1, ..., G_0\})$. Therefore each set $\Lambda_{k,\delta}$ has a finite base. The next lemma shows that to find a base of $\Lambda_{k,\delta}$ it suffices to examine finitely many candidates.

Lemma 6.1. *For any pair of positive integers k and δ there is a base of $\Lambda_{k,\delta}$ that is a subset of the set $\{G_1, ..., G_9\}$.*

Proof. Suppose that S is a base of $\Lambda_{k,\delta}$.

Claim 12. *No graph in S is a line graph.*

If not, there exists a $G \in S$ which is the line graph of a graph H. Let $t = \max\{k/2+2,\ \delta/2+2,\ |V(H)|\}$. Since H is a subgraph of the complete graph K_t, the graph $G = L(H)$ is an induced subgraph of $L(K_t)$.

Since each edge-cut in K_t that separates two edges has at least $2(t-2)$ elements, the graph $L(K_t)$ is $2(t-2)$-regular and $2(t-2)$-connected. Hence $L(K_t) \in \Lambda_{k,\delta} - F_{k,\delta}(S)$, contrary to the assumption that S is a base of $\Lambda_{k,\delta}$. This proves Claim 12.

Now let S be a base of $\Lambda_{k,\delta}$ with $|S \backslash \{G_1, ..., G_9\}|$ minimized. If $|S \backslash \{G_1, ..., G_9\}| = 0$, then we are done. Therefore we assume that there exists a graph $G \in S \backslash \{G_1, ..., G_9\}$. By Claim 12, G is not a line graph. By Theorem 1.1, the graph G contains a G_i as an induced subgraph for some i with $1 \leqslant i \leqslant 9$. Set

$$S^* = \begin{cases} (S \backslash \{G\} \cup \{G_i\}) & \text{if } \quad G_i \notin S \\ S \backslash \{G\} & \text{if } \quad G_i \in S. \end{cases}$$

Since no graph in S^* is a line graph, we have $\Lambda_{k,\delta} \subseteq F_{k,\delta}(S^*)$. By the definition of S^* we have $F_{k,\delta}(S^*) \subseteq F_{k,\delta}(S)$. Since $\Lambda_{k,\delta} = F_{k,\delta}(S)$, the previous containments give $\Lambda_{k,\delta} = F_{k,\delta}(S^*)$. Thus the set S^* is a base of $\Lambda_{k,\delta}$. Then either $|S^* \backslash \{G_1, ..., G_9\}| < |S \backslash \{G_1, ..., G_9\}|$, or $|S^*| < |S|$, a contradiction. ▮

Lemma 6.2. *Let $S \subseteq \{G_1, ..., G_9\}$ be a base of Λ_k. Then*

(1) *S contains $G_1 = K_{1,3}$ and $G_2 = K_5 - e$.*

(2) *S contains G_3 or G_4.*

(3) *S contains G_3 or G_6.*

Proof. (1) If $K_{1,3} \notin S$ then each n-connected triangle-free graph (for $n \geqslant \max\{3, k, \delta\}$) contains only the claw as an induced subgraph among the graphs $G_1, ..., G_9$. Thus all triangle-free n-connected graphs belong to the set $F_{k,\delta}(S) \backslash \Lambda_{k,\delta}$, contrary to the assumption that $F_{k,\delta}(S) = \Lambda_{k,\delta}$.
Similarly, if $K_5 - e \notin S$, then for sufficiently large n the graph $K_n - e$ belongs to $F_{k,\delta}(S) \backslash \Lambda_{k,\delta}$, a contradiction. Thus $\{K_{1,3}, K_5 - e\} \subseteq S$.

(2) For an integer $n \geqslant \max\{5, k+1, \delta\}$, construct the graph H_n by joining two copies of K_n with the vertex sets $\{a_1, ..., a_n\}$ and $\{b_1, ..., b_n\}$ by the matching $\{a_i b_i \mid i = 1, ..., n\}$ and an extra edge $a_1 b_2$ (see Fig. 4 for the graph H_5). Then the graph H_n is k-connected and has minimum degree at least δ, and it contains only G_3 and G_4 as induced subgraphs among the graphs $G_1, ..., G_9$. Namely $\langle a_5, a_1, b_1, b_2, b_3, b_4 \rangle \cong G_3$ and $\langle a_3, b_3, b_1, b_2, a_1 \rangle \cong G_4$. One can verify that H_n does not contain copies of the other graphs from the set $\{G_1, ..., G_9\}$ by using the fact that each of the graphs $G_2, ..., G_9$ contains an induced copy of $K_4 - e$ and each induced copy of $K_4 - e$ in H_n contains the edge $a_1 b_2$. Since the vertices of H_n can be covered by two complete graphs, H_n is claw-free. Hence if neither G_3 nor G_4 belongs to S then $H_n \in F_{k,\delta}(S) - \Lambda_{k,\delta}$, a contradiction. Therefore S contains G_3 or G_4.

(3) For $n \geqslant \max\{k, \delta, 2\}$ construct the graph H'_n consisting of three complete graphs K_{n+1}, K_{n+2}, and K_{2n} with the vertex sets $\{a_0, a_1, ..., a_n\}$, $\{a_{n+1}, ..., a_{2n+2}\}$ and $\{b_1, ..., b_{2n}\}$ joined together by the matching $\{a_i b_i \mid i = 1, 2, ..., 2n\}$ and the edges $a_0 a_{2n+2}$ and $a_0 a_{2n+1}$ (see Fig. 4 for H'_4).

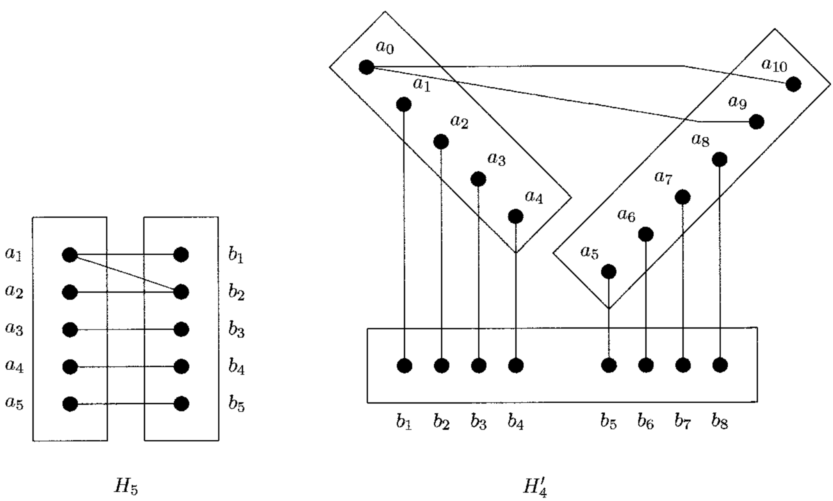


FIG. 4. The graphs H_5 and H'_4 constructed in Section 2. The rectangles represent complete subgraphs.

Then $\langle a_1, a_0, a_{2n+2}, a_{2n+1}, a_{2n}, a_{2n-1}\rangle \cong G_3$ and $\langle a_1, a_0, a_{2n+2}, a_{2n+1}, a_{2n}, b_{2n}\rangle \cong G_6$.

Note that each induced copy of $K_4 - e$ in H'_n contains the triangle $a_0 a_{2n+1} a_{2n+2}$. Using this one can check that the graph H'_n is k-connected and has minimum degree at least δ, and it contains only G_3 and G_6 as induced subgraphs among the graphs $G_1, ..., G_9$. Thus the set S contains G_3 or G_6. ∎

It follows by these two lemmas that each base of $\Lambda_{k,\delta}$ contains at least three graphs. Now we will prove that if $\Lambda_{k,\delta}$ can be characterized by three graphs then $\Lambda_{k,\delta} \subseteq \Lambda_{3,7}$.

Suppose that $\Lambda_{k,\delta}$ can be characterized by three graphs. Then, according to Lemma 6.1, the three graphs can be chosen from the set $\{G_1, ..., G_9\}$ and by the last claim they must be $K_{1,3}$, $K_5 - e$ and G_3. Therefore $\Lambda_{k,\delta} = F_{k,\delta}(\{K_{1,3}, K_5 - e, G_3\})$. A dumbbell consisting of two copies of K_n, for $k \geqslant 4$, sharing a common edge is a 2-connected $\{K_{1,3}, K_5 - e, G_3\}$-free graph with the minimum degree $n-1$. This dumbbell is not a line graph since it contains a copy of G_7. Therefore for all positive integers $k^* \leqslant 2$ and δ we have $\Lambda_{k^*,\delta} \neq F_{k^*,\delta}(\{K_{1,3}, K_5 - e, G_3\})$.

Therefore $k \geqslant 3$. Figure 3 shows a 6-connected 6-regular graph that belongs to the set $F_{k^*,\delta}(\{K_{1,3}, K_5 - e, G_3\}) \setminus \Lambda_{k^*,\delta}$ for all $k^* \leqslant 6$ and $\delta \leqslant 6$. Thus we have $k \geqslant 7$ or $\delta \geqslant 7$. If $k \geqslant 7$ then $\Lambda_{k,\delta} \subseteq \Lambda_{3,7}$. Otherwise $\delta \geqslant 7$ and $\Lambda_{k,\delta} \subseteq \Lambda_{3,7}$, since $k \geqslant 3$. In both cases $\Lambda_{k,\delta} \subseteq \Lambda_{3,7}$.

It follows from Corollary 1.4 that $\Lambda_{3,7}$ and thus any of its subsets of the form $\Lambda_{k,\delta}$ can be characterized by three graphs. This completes the proof of Corollary 1.5. ∎

7. HAMILTONIAN LINE GRAPHS

A remarkable connection between the line graphs and the claw-free graphs has been found by Ryjáček.

Theorem 7.1 (Ryjáček [19]). *For each positive integer k the following two statements are equivalent:*

(1) *Each k-connected claw-free graph is hamiltonian.*

(2) *Each k-connected line graph is hamiltonian.*

Thus the well known conjecture due to Matthews and Summer [16] asserting that every 4-connected claw-free graph is hamiltonian is equivalent with Thomassen's conjecture [21] asserting that every 4-connected line graph is hamiltonian. Simin Zhan verified Thomassen's conjecture for

7-connected line graphs. Li [15] and Brandt [5] have made some progresses in different directions.

THEOREM 7.2 (Zhan [22]). *Each 7-connected line graph is hamiltonian-connected.*

THEOREM 7.3 (Li [15]). *Every* 6-*connected claw-free graph with at most* 33 *vertices of degree* 6 *is hamiltonian.*

THEOREM 7.4 (Brandt [5]). *Every* 9-*connected claw-free graph is hamiltonian connected.*

Pairs of forbidden induced subgraphs sufficient to imply various hamiltonian type properties wee also studied by Bedrossian [1], Broersma and Veldman [6], Duffus *et al.* [10], Faudree and Gould [11], and others.

Ryjáček used Zhan's theorem to show that each 7-connected claw-free graph is hamiltonian. It seems that Ryjáček's technique cannot be used to show that each 7-connected claw-free graph is hamiltonian-connected. We believe that in order to improve Zhan's result it is helpful to understand the structure of 7-connected line graphs. We show that these graphs can be characterized by forbidding only the three induced subgraphs $K_{1,3}$, $K_5 - e$ and G_3, although Beineke's characterization of all line graphs required nine forbidden induced subgraphs. Then we obtain:

THEOREM 7.5. *Each 7-connected* $\{K_{1,3}, K_5 - e, G_3\}$-*free graph is hamiltonian-connected.*

Conjecture 7.6. Each 7-connected claw-free graph is hamiltonian-connected.

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